THE PRINCIPLE OF TRANSFERENCE BETWEEN REAL AND DUAL LORENTZIAN SPACES AND DUAL LORENTZIAN ANGLES

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Dual number algebra is powerful mathematical tool for the computer-aided geometrical design (CAGD), the kinematic and dynamic analysis of spatial mechanism, robotic and human body motion analysis.

In this study we define the fundamental concepts of the dual Lorentzian space, the principle of transference and the eight dual Lorentzian angles between oriented timelike, spacelike and lightlike lines in the 3-dimensional Lorentz space R_1^3. According to this theory; $\tilde{H}_0^2$ is the new dual model of the hyperbolic geometry from non–Euclidean geometries in the dual Lorentzian space D_1^3 and it contains the real unit hyperbolic sphere H_0^2 (real hyperbolic plane or real hyperboloid model). The unit dual Lorentzian sphere $\tilde{S}_1^2$ is dual de-Sitter Space-Time containing the real de-Sitter Space-Time S_1^2 in the 3- dimensional Lorentz space R_1^3. Also, the dual lightlike cone $\tilde{\Lambda}^2$ contains real lightlike cone Λ^2. The dual Lorentzian space and the principle of transference are power tools for the geometries of the curves (Lorentzian ruled surfaces) on the dual Lorentzian quadrics $\tilde{H}_0^2$, $\tilde{S}_1^2$, $\tilde{\Lambda}^2$; the computer aided Lorentzian geometric design (CALGD), the dual Lorentzian spherical kinematic (DLSK), dual hyperbolic spherical kinematic (DHSK), dual lightlike cone kinematic (DLCK), dual Lorentzian spatial kinematic (DLSK), the dual Lorentzian spatial mechanism (DLSM), Dual Lorentzian robotic (DLR), and workers on the theories of special and general relativity.

We hope that this work not only concerns Lorentz geometry and Relativity workers but also directly related to astronomy with many fields of engineering.

Key words:

Dual Lorentzian space, The Principle of Transference, Unit dual Lorentzian sphere, Unit dual hyperbolic Sphere, Dual Lightlike cone, Dual Lorentzian Angles.

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Fundamental Theorems:

Theorem 3.5 Let $\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}} \in \tilde{S}_1^2 \cup \tilde{H}^+ \cup \tilde{\Lambda}^+$. The norm of the *Lorentzian cross product* of $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{b}}$ is given by

1. $\left|\tilde{\boldsymbol{a}}\times\tilde{\boldsymbol{b}}\right| = \sinh\overline{\theta}$ if $\tilde{\boldsymbol{a}},\, \tilde{\boldsymbol{b}} \in \tilde{H}^+$,

2. **i)** $\left|\tilde{\boldsymbol{a}}\times\tilde{\boldsymbol{b}}\right| = \sin\overline{\theta}$ if $\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}} \in \tilde{S}_1^2$ and the planes $\tilde{\boldsymbol{a}}^\perp$ and $\tilde{\boldsymbol{b}}^\perp$ intersect in $\tilde{H}^+$,

ii) $\left|\tilde{\boldsymbol{a}}\times\tilde{\boldsymbol{b}}\right| = \sinh\overline{\theta}$ if $\tilde{a},\, \tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}} \in \tilde{S}_1^2$ and the planes $\tilde{\boldsymbol{a}}^\perp$ and $\tilde{\boldsymbol{b}}^\perp$ do not intersect in $\tilde{H}_c^2 = \tilde{H}^+ \cup \partial\tilde{H}^+$,

iii) $\left|\tilde{\boldsymbol{a}}\times\tilde{\boldsymbol{b}}\right| = 0$ if $\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}} \in \tilde{S}_1^2$ and the planes $\tilde{\boldsymbol{a}}^\perp$ and $\tilde{\boldsymbol{b}}^\perp$ do not intersect in $\tilde{H}^+$ but intersect in $\partial\tilde{H}^+$,

3. $\left|\tilde{\boldsymbol{a}}\times\tilde{\boldsymbol{b}}\right| = \cosh\overline{\theta}$ if $\tilde{\boldsymbol{a}} \in \tilde{H}^+$ and $\tilde{\boldsymbol{b}} \in \tilde{S}_1^2$,

4. $\left|\tilde{\boldsymbol{a}}\times\tilde{\boldsymbol{b}}\right| = e^{-\overline{d}}$ if $\tilde{\boldsymbol{a}} \in \tilde{S}_1^2$ and $\tilde{\boldsymbol{b}} \in \tilde{\Lambda}^+$,

5. $\left|\tilde{\boldsymbol{a}}\times\tilde{\boldsymbol{b}}\right| = e^{-\overline{d}}$ if $\tilde{\boldsymbol{a}} \in \tilde{H}^+$ and $\tilde{\boldsymbol{b}} \in \tilde{\Lambda}^+$,

6. $\left|\tilde{\boldsymbol{a}}\times\tilde{\boldsymbol{b}}\right| = 2e^{-\overline{d}}$ if $\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}} \in \tilde{\Lambda}^+$.

Theorem 4.1 (***The Principle of Lorentzian Transference***)

The unit dual timelike (spacelike) vectors of the sphere $\tilde{H}_0^2$ $(\tilde{S}_1^2)$ and the dual lightlike vectors of the dual lightlike cone $\tilde{\Lambda}^2$ are in one-to-one correspondence with oriented timelike (spacelike) lines and the oriented lightlike lines of the Lorentzian lines space IR_1^3, respectively.

Theorem 4.2 (*Dual Lorentzian Angles*)

The Lorentzian inner product of two vectors $\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}} \in \tilde{S}_1^2 \cup \tilde{H}^+ \cup \tilde{\Lambda}^+$ can be interpreted as follows:

a) If $\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}}$ are positive (negative) dual vectors in $\tilde{H}^+$ the *dual hyperbolic angle* between them is dual number $\tilde{\theta} = \theta + \varepsilon\,\theta^*$ given by

$$\langle\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}}\rangle = -\cosh\overline{\theta}. \tag{4.10}$$

b) If $\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}} \in \tilde{S}_1^2$ the one of the following holds:

i) Two dual geodesic planes $\tilde{\boldsymbol{a}}^\perp$ and $\tilde{\boldsymbol{b}}^\perp$ intersect in $\tilde{H}^2$ if and only if $\left|\langle\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}}\rangle\right| \langle 1$. In this case are positive (negative) dual vectors in $\tilde{H}^+$ the *dual spacelike angle* between them, measured in $\pi_{\tilde{\boldsymbol{a}}}$ and $\pi_{\tilde{\boldsymbol{b}}}$, is the dual number $\tilde{\theta} = \theta + \varepsilon\,\theta^*$ given by

$$\langle\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}}\rangle = \cos\overline{\theta}. \tag{4.11}$$

ii) Two dual geodesic planes $\tilde{\boldsymbol{a}}^{\perp}$ and $\tilde{\boldsymbol{b}}^{\perp}$ do not intersect in $\tilde{H}^2 = \tilde{H}^2 \cup \partial \tilde{H}^2$ if and only if $\left|\langle \tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}} \rangle\right| \rangle 1$. In this case the *dual central angle* between them is the dual number $\tilde{\theta} = \theta + \varepsilon \theta^*$ given by

$$\left|\langle \tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}} \rangle\right| = \cosh \overline{\theta}. \tag{4.12}$$

iii) Two dual geodesic planes $\tilde{\boldsymbol{a}}^{\perp}$ and $\tilde{\boldsymbol{b}}^{\perp}$ do not intersect in $\tilde{H}^2$ but intersect in $\partial \tilde{H}^2$ if and only if

$$\left|\langle \tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}} \rangle\right| = \cos \overline{0} = 1. \tag{4.13}$$

In this case the *dual Lorentzian angle* between them is the zero dual number $\overline{0}$.

c) Let $\tilde{\boldsymbol{a}} \in \tilde{H}^+$ and $\tilde{\boldsymbol{b}} \in \tilde{S}_1^2$. The *dual timelike angle* between them is dual number $\tilde{\theta} = \theta + \varepsilon \theta^*$ given by

$$\left|\langle \tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}} \rangle\right| = \sinh \overline{\theta}. \tag{4.14}$$

d) **i)** Let $\tilde{\boldsymbol{a}} \in \tilde{S}_1^2$ and $\tilde{\boldsymbol{b}} \in \tilde{\Lambda}^2$. The *first dual lightlike angle* between them is dual number $\overline{d} = d + \varepsilon d^*$ given by

$$\left|\langle \tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}} \rangle\right| = e^{\overline{d}}. \tag{4.15}$$

ii) Let $\tilde{\boldsymbol{a}} \in \tilde{H}^+$ and $\tilde{\boldsymbol{b}} \in \tilde{\Lambda}^+$. Then the *second dual lightlike angle* between them is dual number $\overline{d} = d + \varepsilon d^*$ given by

$$\langle \tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}} \rangle = -e^{\overline{d}}. \tag{4.16}$$

iii) Let $\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}} \in \tilde{\Lambda}^+$. The *third dual lightlike angle* between them is dual number $\overline{d} = d + \varepsilon d^*$ given by

$$\left|\langle \tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}} \rangle\right| = -2e^{\overline{d}}. \tag{4.17}$$